Spinor Couplings to Dilaton Gravity induced by the Dimensional Reduction of Topologically Massive Gravity

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19 January 2004

Abstract

A Dirac spinor is coupled to topologically massive gravity and the $D = 3$ dimensional action is reduced to $D = 2$ dimensions with a metric that includes both the electromagnetic potential 1-form A and a dilaton scalar ϕ. The reduced, electrically charged spinor is made a mass eigenstate with a (local) chiral rotation. The non-trivial interactions thus induced are discussed.

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1 Introduction

Two-dimensional dilaton gravity theories coupled to matter have attracted a lot of attention because of their relevance to black holes in effective string models and also because for some particular choices of the dilaton and matter couplings they may be related with higher dimensional theories of gravity. In the standard Kaluza-Klein theories one usually starts with the Einstein-Hilbert action for gravity in $D > 2$ and obtains a reduced Lagrangian density in 2-dimensions by inserting the unique D-dimensional metric-compatible, torsion-free (Levi-Civita) connection into the original Lagrangian D-form. This connection is calculated from a special choice of D-dimensional metric that includes components of vector fields (identified with gauge potentials) and scalar fields (dilatons) together with the 2-dimensional metric. Many generalizations that involve higher order curvature invariants in the action and/or connections with both torsion and non-metricity exist in the literature [1],[2]. We will concentrate our attention here to $D = 3$ dimensions. It is well-known that Einstein's theory in 3-dimensions has no propagating degrees of freedom and no Newtonian limit. A physically interesting modification of the theory is provided by the addition of the gravitational Chern-Simons term to the action. With this addition, new degrees of freedom are introduced and one now has a dynamical theory that is called topologically massive gravity [3], [4]. Recent work on the Kaluza-Klein reduction of this theory to $D = 2$ dimensions produced an interesting dilaton gravity theory and prompted further research [5],[6],[7]. However, there are not many papers that discuss fermionic matter couplings to dilaton gravity [8]. In this paper we couple a Dirac spinor in $D = 3$ dimensions to topologically massive gravity. The subsequent Kaluza-Klein reduction to $D = 2$ dimensions produce non-trivial interactions that we discuss. We analyze our model in terms of coordinate independent concepts and make extensive use of the calculus of exterior forms.

2 Dimensional reduction from $D = 3$ to $D = 2$

On a 3-dimensional manifold M_3 we denote by $\{\mathbb{X}_A\}$, $A = 0, 1, 2$, an arbitrary 3-frame and the dual co-frame $\{E^A\}$ by

$$E^A(\mathbb{X}_B) = \delta^A_B \; . \tag{1}$$

Frames are declared orthonormal with respect to the general metric

$$\mathbb{G} = \eta_{AB} E^A \otimes E^B, \tag{2}$$

where $\eta_{AB} = \mathrm{diag}(-,+,+)$. This choice of $\mathbb{G}$ is conserved under local $SO(1,2)$ frame transformations. The metric-compatible connection 1-forms on M_3 satisfy $\Omega_{AB} = -\Omega_{BA}$. The torsion

and curvature forms associated with these co-frame and connection forms are given by the structure equations:

$$\mathbb{T}^A = dE^A + \Omega^A{}_B \wedge E^B, \tag{3}$$

$$\mathbb{R}^A{}_B = d\Omega^A{}_B + \Omega^A{}_C \wedge \Omega^C{}_B. \tag{4}$$

The manifold M_3 is oriented with the volume 3-form

$$^{\#}1 = E^0 \wedge E^1 \wedge E^2 \tag{5}$$

where $^{\#}$ denotes the Hodge map and it is convenient to employ in the following the graded interior operators $\imath_{\mathbb{X}_A}$:

$$\imath_{\mathbb{X}_A} E^B = \delta^B_A \ . \tag{6}$$

Spinors associated with M_3 are defined as complex vectors carrying a representation of the covering group of $SO(1,2)$. To make contact with the spinors associated with the covering of the Lorentz group $SO(1,1)$, we enumerate the Clifford algebra generators Γ_A associated with the orthonormal frames of M_3 satisfying

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} \tag{7}$$

with the choice

$$\Gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad \Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \Gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{8}$$

The generators of the Clifford algebra associated with the orthonormal frames of space-time M_2 are

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab} \quad , \quad a = 0, 1 \tag{9}$$

where $\eta_{ab} = \mathrm{diag}(-, +)$. The algebra of $SO(1,2)$ is generated by

$$\Sigma_{AB} = \frac{1}{4}[\Gamma_A, \Gamma_B] \tag{10}$$

and that of $SO(1,1)$ by

$$\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b] \ . \tag{11}$$

A 2-component complex spinor Ψ in $D = 3$ dimensions is constructed locally from a set of two complex functions on M_3. It will be reduced in a certain way below to a 2-component complex spinor ψ on M_2.

Three local co-ordinate functions for M_3 will be labelled by $x^A \equiv (x^a, y)$. The submanifold $y = 0$ in this chart is then M_2 which inherits a patch with two co-ordinates (x^0, x^1). Now we write Kaluza-Klein metric as

$$\mathbb{G} = g + \phi^2 A \otimes A + \phi^2(dy \otimes A + A \otimes dy) + \phi^2 dy \otimes dy \tag{12}$$

where the metric on M_2 is

$$g = -e^0 \otimes e^0 + e^1 \otimes e^1 \ . \tag{13}$$

A particularly convenient choice of the orthonormal 3-frame is

$$\begin{aligned} \mathbb{X}_a &= X_a - (\imath_{X_a} A)\partial_y \ , && (14) \\ \mathbb{X}_2 &= \frac{1}{\phi(x)}\partial_y \ , && (15) \end{aligned}$$

where $\partial_y = \frac{\partial}{\partial y} \in T(M_3)$ is a vector field, $\phi(x)$ and $A(x)$ are respectively 0 and 1 forms on M_2 and $X_a \in T(M_2)$. The dual co-frame is

$$\begin{aligned} E^a &= e^a(x) \ , && (16) \\ E^2 &= \phi(x)[dy + A(x)] \ , && (17) \end{aligned}$$

where $e^a(X_b) = \delta^a_b$ and $e^a \in \Lambda^1(M_2)$ satisfy the structure equations

$$de^a + \omega^a{}_b \wedge e^b = 0 \ , \ R^a{}_b = d\omega^a{}_b \ . \tag{18}$$

on M_2. The orientation of M_2 is given by the volume element

$$^*1 = e^0 \wedge e^1 \tag{19}$$

where * is the Hodge map. The field $\phi(x)$ simply scales the vector field ∂_y to normalize it with respect to the metric $\mathbb{G}$. Using this reduction scheme we can reduce the Levi-Civita connection 1-forms $\Omega^A{}_B$ as follows:

$$\begin{aligned} \Omega^0{}_1 &= \omega^0{}_1 + \frac{\phi}{2} f E^2 \ , \\ \Omega^0{}_2 &= \frac{\phi}{2} f e^1 - \frac{\partial^0 \phi}{\phi} E^2 \ , \\ \Omega^1{}_2 &= \frac{\phi}{2} f e^0 - \frac{\partial^1 \phi}{\phi} E^2 && (20) \end{aligned}$$

where

$$F = dA = f * 1 \ . \tag{21}$$

The corresponding curvature 2-forms are

$$
\begin{aligned}
\mathbb{R}^0{}_1 &= R^0{}_1 + \frac{3}{4}\phi^2 f^{2*}1 + (\frac{3}{2} f d\phi + \frac{\phi}{2} df) \wedge E^2 , \\
\mathbb{R}^0{}_2 &= \frac{1}{2} d(\phi f) \wedge e^1 - \partial^0 \phi F - [\frac{1}{\phi} D(\omega)(\partial^0 \phi) + \frac{\phi^2}{4} f^2 e^0] \wedge E^2 , \\
\mathbb{R}^1{}_2 &= \frac{1}{2} d(\phi f) \wedge e^0 - \partial^1 \phi F - [\frac{1}{\phi} D(\omega)(\partial^1 \phi) + \frac{\phi^2}{4} f^2 e^1] \wedge E^2 . \qquad (22)
\end{aligned}
$$

We consider the action $I[E, \Omega] = \int_{M_3} \mathbb{L}$ where the Lagrangian 3-form

$$
\mathbb{L} = \kappa \mathbb{R}^A{}_B \wedge^{\#} (E_A \wedge E^B) + \frac{\mu}{2} (\Omega^A{}_B \wedge d\Omega^B{}_A + \frac{2}{3} \Omega^A{}_C \wedge \Omega^C{}_B \wedge \Omega^B{}_A) + \lambda^{\#} 1 \qquad (23)
$$

contains the Einstein-Hilbert term with the coupling constant κ, the gravitational Chern-Simons term with the coupling constant μ written in terms of Levi-Civita connection 1-forms $\Omega^A{}_B$ and a cosmological constant λ. A study of this theory can be found in Ref.[9]. We dimensionally reduce (23) and obtain a reduced Lagrangian 3-form $\mathbb{L} = L \wedge dy$ where

$$
\begin{aligned}
L &= \kappa [\frac{\phi}{2} \mathcal{R} + \frac{\phi^3}{4} f^2]^* 1 + \phi \lambda^* 1 \\
&\quad + \mu [(\frac{\phi^2}{2} f \mathcal{R} + \frac{\phi^4}{2} f^3)^* 1 - 2 f d\phi \wedge^* d\phi - \phi df \wedge^* d\phi] . \qquad (24)
\end{aligned}
$$

that agrees with [5], [6] when ϕ is a constant. $\mathcal{R}$ is the scalar curvature on M_2.

We now consider the Dirac Lagrangian 3-form

$$
\mathbb{L}_D = \mathrm{Her}(i \overline{\Psi}\, {}^{\#}\Gamma \wedge \mathbb{D}\Psi) + i M \overline{\Psi} \Psi\, {}^{\#} 1 \qquad (25)
$$

in terms of the Clifford algebra $\mathcal{C}\ell_{1,2}$-valued 1-forms $\Gamma = \Gamma_A E^A$ and $M = \frac{mc}{\hbar}$ is the Dirac mass. The covariant exterior derivative of a Dirac spinor is given explicitly as

$$
\mathbb{D}\Psi = d\Psi + \frac{1}{2} \Omega^{AB} \Sigma_{AB} \Psi . \qquad (26)
$$

Our reduction ansatz [10] for the spinors is

$$
\begin{aligned}
\Psi(x, y) &= e^{\gamma_5 \lambda(x) + iqy} \psi(x) , \qquad (27) \\
\overline{\Psi}(x, y) &= \Psi^+ \Gamma_0 = \overline{\psi} e^{\gamma_5 \lambda(x) - iqy} \qquad (28)
\end{aligned}
$$

where $\lambda(x)$ is a real function on M_2 to be determined later, q is a constant identified with the electric charge and

$$
\Gamma_a = \gamma_a \quad , \quad \gamma_5 = i \gamma_0 \gamma_1 = i \Gamma_2 . \qquad (29)
$$

Here we assumed that the spinor Ψ is periodic in y in order to obtain a minimal coupling to electromagnetism governed by the electromagnetic 1-form A. Now we first calculate

$$^{\#}\Gamma = -i\gamma_5^* 1 - E^2 \wedge^* \gamma \tag{30}$$

where $\gamma = \gamma_a e^a$ is the Clifford algebra $\mathcal{C}\ell_{1,1}$-valued 1-form and then

$$\begin{aligned}\mathbb{D}\Psi = e^{\gamma_5\lambda(x)+iqy}[D\psi + \gamma_5 d\lambda(x)\psi + E^2 i(\frac{q}{\phi} - \frac{\phi}{4} f\gamma_5)\psi] \\ +e^{-\gamma_5\lambda(x)+iqy}[i\frac{\phi}{4}f^*\gamma\gamma_5 + E^2 i\frac{1}{2\phi}{}^*({}^*\gamma \wedge d\phi)\gamma_5]\psi \ .\end{aligned} \tag{31}$$

Here

$$D\psi = d\psi + \frac{1}{2}\omega^{ab}\sigma_{ab}\psi - iqA\psi \ . \tag{32}$$

is the $U(1)$-covariant exterior derivative of a spinor and physically describes the minimal interactions with gravity and electromagnetism. The second term on the right hand side of (31) simulates a chiral interaction that senses the handedness of the ψ spinor if λ is not a constant. With these definitions, the dimensionally reduced Dirac Lagrangian 2-form becomes

$$\begin{aligned}L_D = \phi\{-\mathrm{Her}(i\overline{\psi}\,{}^*\gamma \wedge D\psi) + i\overline{\psi}\, e^{2\lambda\gamma_5}(\frac{q}{\phi}\gamma_5 + M)\psi^* 1 + id\lambda \wedge \overline{\psi}\,{}^*\gamma\gamma_5\psi \\ + \frac{i}{2\phi}\overline{\psi}\, d\phi \wedge {}^*\gamma\psi - i\frac{\phi}{4} f\overline{\psi}\, e^{2\lambda\gamma_5}\psi^* 1\} \ .\end{aligned} \tag{33}$$

The first term in (33) is the kinetic term, the second is the mass term and the rest are new interaction terms. However, γ_5 in the mass term means that the left and right handed parts of the reduced 2-component spinor would have different masses. Since we wish to identify both the left and the right handed parts of the reduced 2-component spinor with the chiral components of a Dirac particle of definite mass, we set

$$\tan 2\lambda = -\frac{q}{M\phi} \ . \tag{34}$$

Here since ${\gamma_5}^2 = -1$ we write

$$e^{2\lambda\gamma_5} = \cos 2\lambda + \gamma_5 \sin 2\lambda \ . \tag{35}$$

It follows from (34) that

$$d\lambda = \frac{qM}{2(q^2 + M^2\phi^2)} d\phi \ . \tag{36}$$

Thus we write down the reduced Dirac Lagrangian 2-form as

$$\begin{aligned} L_D &= \phi\{-\mathrm{Her}(i\overline{\psi}\,{}^*\gamma \wedge D\psi) - i\mathcal{M}\overline{\psi}\,\psi{}^*1 \\ &\quad + i\frac{1}{2\phi}d\phi \wedge \overline{\psi}\,{}^*\gamma\psi + i\frac{qM}{2\mathcal{M}^2\phi^2}d\phi \wedge \overline{\psi}\,{}^*\gamma\gamma_5\psi \\ &\quad + i\frac{fM\phi}{4\mathcal{M}}\overline{\psi}\,\psi{}^*1 - i\frac{qf}{4\mathcal{M}}\overline{\psi}\,\gamma_5\psi{}^*1\} \end{aligned} \tag{37}$$

where the effective mass is

$$\mathcal{M} = \sqrt{M^2 + \left(\frac{q}{\phi}\right)^2} . \tag{38}$$

3 Conclusion

We have coupled a Dirac spinor Ψ to topologically massive gravity and dimensionally reduced the $D = 3$ dimensional action to $D = 2$ dimensions with a Kaluza-Klein metric that includes both the electromagnetic potential 1-form A and a dilaton scalar ϕ. The reduced, electrically charged spinor ψ becomes a mass eigenstate after a (local) chiral rotation with (34). The Lagrangian 2-form (37) summarizes all the non-trivial interactions thus induced. In particular the effective mass $\mathcal{M}$ depends on the 3-dimensional mass M, the electric charge q as well as the dilaton field ϕ. Even if we initially set M to zero in 3-dimensions, there may still be a non-zero effective mass induced in 2-dimensions with $\lambda = -\frac{\pi}{4}$ provided $q \neq 0$. There are interactions that describe vector and pseudo-vector couplings to the gradient of the dilaton field. Finally, the last two terms in (37) describe non-minimal electric and magnetic dipole moment interactions, respectively. They provide additional contributions to the established gyromagnetic ratio defined by the minimal electromagnetic coupling to the spinor. It is interesting to note that both the dipole moments have explicit dependence on the dilaton field. We intend to investigate solutions to the induced Dirac equation in a separate paper.

Acknowledgement

One of the authors (MA) is supported by the Scientific Research Project (BAP) 2002FEF007, Pamukkale University, Denizli, Turkey.

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